Last class:

Theorem (25.4 in book): $S \subset \mathbb{R}$

$$f_n : S \to \mathbb{R} \qquad n = 0, 1, 2, \ldots$$

If (f_n) is uniform Cauchy

$\Rightarrow \exists$ function $f : S \to \mathbb{R}$ s.t.

$f_n \to f$ uniformly.

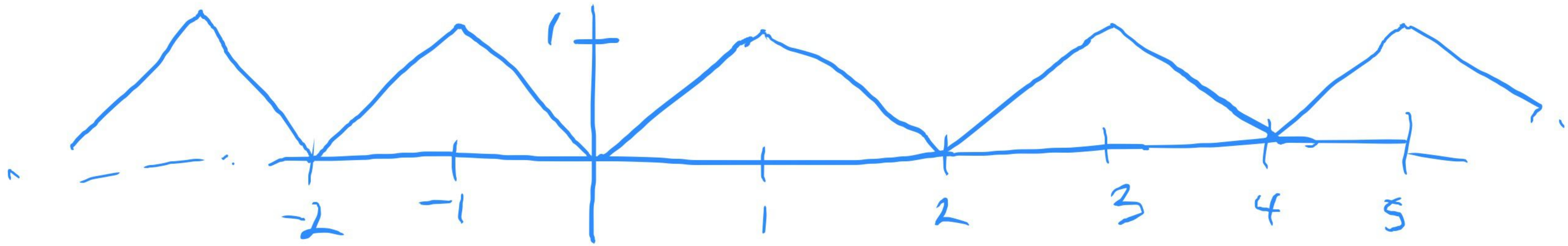
Last class: **M-test** If $g_n : S \to \mathbb{R}$ functions

$$|g_n(x)| \le M_n \qquad \forall x \in S$$

If $\sum_{n=0}^{\infty} M_n < \infty \Rightarrow \sum_{n=0}^{\infty} g_n(x)$ converges to a function $f(x)$

Example 1 (Example 3 on p 204 in book)

$g(x)$:



Define $\tilde{g}_k(x) = g(4^k x)$

$\tilde{g}_1(x) = g(4x)$

Define $\boxed{g_k(x) = \left(3/4\right)^k \tilde{g}_k(x)} \Rightarrow \boxed{|g_k(x)| \le \left(\frac{3}{4}\right)^k \cdot 1}$

We know: $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} (3/4)^n = \frac{1}{1-3/4} = 4$

$\Rightarrow$ can apply M-test

Theorem $\Rightarrow$ $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly! to a continuous function $f(x)$

(cont. because the g_n's are continuous and convergence is uniform).

One can show:

While this function $f(x)$ is continuous, it is NOT differentiable at any point $x \in \mathbb{R}$!

Example 2

Consider $\sum_{k=0}^{\infty} 2^{-k} x^k$

Claim: (a) This power series converges uniformly on the interval $[-R_1, R_1]$ for any $0 < R_1 < 2$.

(b) power series converges to a continuous function on $(-2,2)$ (open interval)

proof want to apply M-test!

$g_k(x) = 2^{-k} x^k = (x/2)^k$

$x \in [-R_1, R_1]$

$\Rightarrow |x| \leq R_1 < 2 \quad | \; \frac{1}{2}$

$\Rightarrow \frac{|x|}{2} \leq \frac{R_1}{2} < 1$

$\Rightarrow |g_k(x)| = |x/2|^k$

$\leq |R_1/2|^k = M_k$

$\sum M_k = \sum (R_1/2)^k = \frac{1}{1 - R_1/2} < \infty$

$\Rightarrow$ can apply M-test:

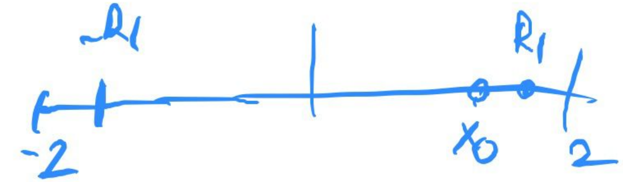
$\Rightarrow$ power series converges uniformly to a continuous function on $[-R_1, R_1]$

(again: The function $f_n = \sum_{k=0}^{n} 2^{-k} x^k$ is continuous (it is a polynomial!)

uniform convergence $\Rightarrow$ $\sum_{k=0}^{\infty} 2^{-k} x^k = \lim_{n\to\infty} f_n$ is continuous)

(b) let $x_0 \in (-2, 2)$

$\Rightarrow$ $|x_0| < 2$



let $\boxed{R_1 = \frac{|x_0| + 2}{2}}$ (could be any number between $|x_0|$ and 2)

by part (a): power series converges for $x_0 \in [-R_1, R_1]$, $R_1 < 2$!

convergence on $[-R_1, R_1]$ is uniform!

$\Rightarrow$ Limit function is continuous at x_0

true for any $x_0 \in (-2,2)$

$\Rightarrow$ Limit function is cont. on $(-2,2)$.

<u>Remark</u> For our specific example, we have an explicit expression for the limit function:

$$\sum_{k=0}^{\infty} (x/2)^k \underset{\substack{\uparrow \\ \text{formula for geometric series}}}{=} \frac{1}{1-x/2} = \frac{2}{2-x}$$

Ch. 26

Example 2 is a special case of a much more general result! Essentially it shows that we get a continuous function from <u>any</u> power series within its radius of convergence.

More precisely:

<u>Theorem</u> Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $0 < R \leq \infty$.

If $0 < R_1 < R$, then the power series converges uniformly to a continuous function on $[-R_1, R_1]$.

(Remark: Example 2 is a special case with $R=2$ and $a_n = 2^{-n}$)

Proof. Recall: radius of convergence R was defined by

$$\frac{1}{R} = \beta = \limsup_{n\to\infty} |a_n|^{1/n}$$

$\Rightarrow$ $\sum a_n x^n$ and $\sum |a_n| x^n$ have same radius of convergence.

Let $0 < R_1 < R$ $\Rightarrow$ $\displaystyle\sum_{n=0}^{\infty} \underbrace{|a_n| R_1^n}_{M_n} < \infty$

$\Rightarrow$ can apply M-test for $|x| \leq R_1$

$\Rightarrow$ $\displaystyle\sum_{n=0}^{\infty} \underbrace{a_n x^n}_{g_n(x)}$ converges uniformly to a continuous function on $[-R_1, R_1]$

(because $|a_n x^n| \leq M_n = |a_n R_1^n|$). $\Rightarrow$ claim!

Remark: This is a useful theorem for many applications!

E.g. $\underbrace{\text{for}}_{\text{many}}$ differential equations one can only obtain the solutions as a power series for which we do not know any simpler expression.

Example: Bessel functions:

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\, n!\, (n+k)!} x^{2n+k}$$

here: radius of convergence $= \infty$